A Simple Presentation of the Siegel Modular Groups

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ABSTRACT

Given is a presentation for the Siegel modular group $\mathrm{Sp}(2g,\mathbb{Z})$ with at most three generators and $3g+5$ relations.

INTRODUCTION

The *Siegel modular groups*, also called the *simplectic modular groups*, defined by

$$\mathrm{Sp}(2g,\mathbb{Z})=\left\{\begin{pmatrix} A & S \\ T & B\end{pmatrix}\in\mathscr{M}_{2g\times 2g}(\mathbb{Z});\ AS^t=SA^t,\ BT^t=TB^t,\ AB^t-ST^t=I\right\},$$

for $g\geqslant 1$, are known to be isomorphic to the quotient of the surface mapping class group $\mathscr{M}_g$ modulo the Torelli subgroup $\mathscr{I}_g$, i.e.,

$$\mathrm{Sp}(2g,\mathbb{Z})\cong\mathscr{M}_g/\mathscr{I}_g.$$

As an application of the author's presentations of the surface mapping class

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groups [2–4], the following very simple presentations of the Siegel modular groups will be proved in this paper.

THEOREM 1.

$$\mathrm{Sp}(2,\mathbb{Z}) = \left\langle L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}; (LN)^2 = N^3, N^6 = 1 \right\rangle.$$

THEOREM 2.

$$\mathrm{Sp}(4,\mathbb{Z}) = \Big\langle L, N;\ N^6 = (LN)^5 = (L\overline{N})^{10} = (L\overline{N}LN)^6 = 1,$$

$$L \leftrightarrow N^2LN^4,\ N^3LN^3,\ (L\overline{N})^5 \Big\rangle,$$

where

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad N = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

THEOREM 3. *The group* $\mathrm{Sp}(2g,\mathbb{Z})$, *for* $g \geqslant 3$, *has a presentation of three generators*: L, N, *and* T; *and* $3g+5$ *defining relations*:

$$L \leftrightarrow T^iL\overline{T}^i, T^jN\overline{T}^j, (L\overline{N})^5, N^2\overline{T}NL\overline{N}T\overline{N}^2,$$

$$N \leftrightarrow T^kN\overline{T}^k, TL\overline{T} = N^3LN^3, T\overline{N}LN\overline{T} = N^2L\overline{N}^2,$$

$$N^6 = (LN)^5 = (L\overline{N})^{10} = (L\overline{N}LN)^6 = T^g = (TN^3)^{g-1} = 1,$$

$$N^3TK\overline{T}N^3 = TK\overline{T}\cdot\overline{K}\cdot T\overline{K}\overline{T}\cdot K, \qquad \textit{where} \quad K = NLN^2\overline{L}N^2\overline{L}N,$$

for $1 \leqslant i \leqslant g-1$, $1 \leqslant j \leqslant g-2$, *and* $2 \leqslant k \leqslant g-2$. *Here*

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ & & & & 1 & 0 & \cdots & 0 \\ & & 0 & & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 0 & \cdots & 0 & 1 & & & & \\ 1 & \cdots & 0 & 0 & & & & \\ \vdots & \ddots & \vdots & \vdots & & & 0 & \\ 0 & \cdots & 1 & 0 & & & & \\ & & & & 0 & \cdots & 0 & 1 \\ & & & & 1 & \cdots & 0 & 0 \\ & 0 & & & \vdots & \ddots & \vdots & \vdots \\ & & & & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

Conventionally, the notation $\alpha \leftrightarrow \beta$ means that α and β commute, i.e., $\alpha\beta = \beta\alpha$. The inverses are denoted by overlining the corresponding words, e.g. $\alpha^{-1} = \bar{\alpha}$, and the group multiplication is the multiplication of matrices.

As a direct consequence of the above theorems, we have the following interesting result:

THEOREM 4. *The group* $\mathrm{Sp}(2g, \mathbb{Z})$ *is generated by some periodic elements, for any* $g \geq 1$. *In particular*, $(LN)^4 = N^6 = 1$ *when* $g = 1$, $(LN)^5 = N^6 = 1$ *when* $g = 2$, *and* $(LN)^5 = N^6 = T^g = 1$ *when* $g \geq 3$.

1. THE SURFACE MAPPING GROUP $\mathscr{M}_g$ AND ITS TORELLI SUBGROUP $\mathscr{I}_g$

A *mapping class* f from the group $\mathscr{M}_g$ is an isotopy class of some orientation-preserving self-homeomorphism of the closed orientable surface $\mathbf{F}_g$ of genus g, which is uniquely interpreted by its induced action on the group $\pi_1(\mathbf{F}_g, O)$, where O is a fixed basepoint in $\mathbf{F}_g$. Precisely, if we choose a set of generators

$$\mathscr{B} = \{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\}$$

of the group $\pi_1(\mathbf{F}_g, O)$ such that

$$\prod_{i=1}^{g} a_i b_i \bar{a}_i \bar{b}_i = 1,$$

the mapping class f may be denoted by

$$f = \left[(a_1)f, (a_2)f, \ldots, (a_g)f, (b_1)f, (b_2)f, \ldots, (b_g)f\right].$$

The mapping class f also induces an automorphism of the first homology group $H_1(\mathbf{F}_g, \mathbb{Z})$ of the surface $\mathbf{F}_g$, a free abelian group of rank $2g$, which is known to be an element of the Siegel modular group. Therefore, there is a natural homomorphism

$$\lambda : \mathscr{M}_g \to \mathrm{Sp}(2g, \mathbb{Z}).$$

The kernel of λ, denoted by $\mathscr{I}_g$, is called the Torelli group. As a well-known fact, the map λ is an epimorphism [5]. Thus, the Siegel modular group $\mathrm{Sp}(2g, \mathbb{Z})$ is isomorphic to the quotient of $\mathscr{M}_g$ modulo $\mathscr{I}_g$. For convenience, we will denote the image of a mapping class in $\mathrm{Sp}(2g,\mathbb{Z})$ under λ by the same notation as itself.

Let L, N, and T be three mapping classes defined by

$$L = \left[a_1b_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g\right],$$

$$N = \begin{cases} \left[b_1, b_1\bar{a}_1\right], & g = 1, \\ \left[x\bar{a}_2b_1, \bar{a}_1xb_2, a_3, \ldots, a_g, \bar{a}_1, \bar{a}_2, b_3, \ldots, b_g\right], & g \geqslant 2, \end{cases}$$

$$T = \left[a_g, a_1, \ldots, a_{g-1}, b_g, b_1, \ldots, b_{g-1}\right].$$

where $x = a_1b_1\bar{a}_1\bar{b}_1a_2b_2\bar{a}_2\bar{b}_2$. We denote a word in the form $\beta\alpha\bar{\beta}$ simply by $\langle\beta\rangle\alpha$. In [2–4] we proved the following presentation of the surface mapping class group $\mathscr{M}_g$:

THEOREM 5.

$$\mathscr{M}_1 = \left\langle L, N;\ (LN)^2 = N^3,\ N^6 = 1\right\rangle,$$

$$\mathscr{M}_2 = \left\langle L, N;\ N^6 = (LN)^5 = (L\bar{N})^{10} = (L\bar{N}LN)^6 = 1,\ L \leftrightarrow N^2LN^4, N^3LN^3\right\rangle.$$

In general, for $g \geqslant 3$, *the mapping class group* $\mathscr{M}_g$ *has a presentation of three generators*: L, N, *and* T, *and* $3g+4$ *relations*:

(I) $L \leftrightarrow \langle T^i\rangle L$, $i = 1,2,\ldots, g-1$,
$L \leftrightarrow \langle T^i\rangle N$, $i = 1,2,\ldots, g-2$,
$L \leftrightarrow N^6$, $L \leftrightarrow \langle \bar{N}T\bar{N}(\bar{L}N)^2\rangle L$,
(II) $N \leftrightarrow \langle T^i\rangle N$, $i = 2,3,\ldots, g-2$,
(III) $\langle T\rangle L = \langle N^3\rangle L$, $\langle T\bar{N}\rangle L = \langle N^2\rangle L$,
(IV) $(LN)^5 = N^6$, $(\bar{L}N)^{10} = N^6$,
(V) $\langle T\rangle L \cdot \langle N^2(\bar{N}LNL\bar{N})^3TN\rangle L \cdot \langle \bar{N}TN\rangle L$
$= \langle \bar{N}\rangle L \cdot \langle N\rangle L \cdot \langle T\bar{N}(\bar{L}N)^2\rangle L \cdot \langle T^2\rangle L$,
(VI) $T^g = 1$, $(N^3T)^{g-1} = (L\bar{N}LN)^{6(g-2)}$,
$(\bar{T}N^3L^2(\bar{N}LNL\bar{N})^4)^{g-1} = 1$.

A set of normal generators of the Torelli group $\mathscr{I}_g$ was first found by Birman [1], and then simplified by Powell [6]. Writing them in terms of the generators L, N, and T, we have the next theorem:

THEOREM 6. *$\mathscr{I}_1 = 1$; $\mathscr{I}_2$ is normally generated by $(L\bar{N})^5(\bar{L}N)^5$; and $\mathscr{I}_g$, $g \geqslant 3$, is normally generated by the elements $(L\bar{N}LN)^6$, N^6, and $(L\bar{N})^5(\bar{L}N)^5$.*

For convenience, we will denote $M = \bar{N}LN$ and $P = LML$. From Theorem 5 it is not hard to see that $LML = MLM$ and $(L\bar{N}LN)^6 = P^4$.

2. THE PROOF OF THEOREMS 1, 2, AND 3

By the discussion in the last section, all we need to show is that the defining relations in Theorem 5 and the normal generators in Theorem 6 are consequences of the defining relations in Theorems 1, 2, and 3. Thus, Theorem 1 is obvious, and Theorem 2 is easy, since the relation $L \leftrightarrow (L\bar{N})^5$ is equivalent to the formula $(L\bar{N})^5(\bar{L}N)^5 = 1$.

For the case of genus $g \geqslant 3$, the only relations which are not so obvious are the last one in (I), the lantern law (V), and the last one in (VI), which will be proved in the next several lemmas. In the first lemma we give several relations which come from the corresponding relations in the surface mapping class group that were shown in [4].

LEMMA 7.

(a) $L \leftrightarrow N^2LN^4$, N^3LN^3, N^4LN^2, $\bar{N}LNL\bar{N}$, $NL\bar{N}LN$;
(b) $(\bar{N}LNL\bar{N})^4 = \bar{L}^2$.

Proof. (a): Exactly the same as was shown in [4].
(b): Repeatedly applying the formulas $L \leftrightarrow (L\bar{N})^5$, $(L\bar{N})^{10} = 1$ and those in part (a), we have

$$
\begin{aligned}
(\bar{N}LNL\bar{N})^4 &= (\bar{N}L)NL(\bar{N}^2LN^2)\bar{N}L\bar{N}^2L\bar{N}(N^2L\bar{N}^2)LN(L\bar{N}) \\
&= (\bar{N}L)^2N^2L\bar{N}L(\bar{N}^2L\bar{N}LN^3)\bar{N}(L\bar{N})^2 \\
&= (\bar{N}L)^2N^2L(\bar{N}^3L\bar{N}LN^4)\bar{N}(L\bar{N})^3 \\
&= (\bar{N}L)^4N^4(L\bar{N})^4 \\
&= \bar{L}(L\bar{N})^5N^6(\bar{N}L)^5\bar{L} \\
&= \bar{L}(N\bar{L})^5(\bar{N}L)^5\bar{L} = \bar{L}^2. \qquad \blacksquare
\end{aligned}
$$

Using the formula in Lemma 7(b) and the formula $(TN^3)^{g-1}=1$, the last relation in (VI) is clear.

LEMMA 8. *Denote*

$$K = NLN^2\overline{L}N^2\overline{L}N = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & \cdots & 0 & & & & & \\ -1 & 1 & 0 & \cdots & 0 & & & & & \\ 0 & 0 & 1 & \cdots & 0 & & & 0 & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & & & & & \\ \hline & & & & & 1 & 1 & 0 & \cdots & 0 \\ & & & & & 0 & 1 & 0 & \cdots & 0 \\ & & 0 & & & 0 & 0 & 1 & \cdots & 0 \\ & & & & & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & 0 & \cdots & 1 \end{array}\right).$$

The lantern law is equivalent to the following formula:

$$N^3TK\overline{T}N^3 = TK\overline{T}\cdot\overline{K}\cdot T\overline{K}\overline{T}\cdot K.$$

Proof. For convenience, we will denote

$$K_{12}=K,\qquad K_{23}=TK\overline{T},\qquad K_{13}=N^3TK\overline{T}N^3,$$

$$L_2=TL\overline{T},\qquad M_2=TM\overline{T},\quad \text{and}\quad L_3=T^2L\overline{T}^2.$$

Thus,

$$K = NLN^2\overline{L}N^2\overline{L}N = NL\overline{N}\,\overline{L}_2\overline{M},$$

and the formula claimed in the lemma is equivalent to

$$K_{13}=K_{23}\cdot\overline{K}_{12}\cdot\overline{K}_{23}\cdot K_{12},\qquad \text{or}\quad K_{13}\cdot\overline{K}_{12}\cdot K_{23}\cdot K_{12}\cdot\overline{K}_{23}=1.$$

On the other hand, we recall the lantern law:

$$\langle T\rangle L\cdot\langle N^3\overline{L}\overline{M}L^2TN\rangle L\cdot\langle\overline{N}TN\rangle L$$

$$\cdot\langle\overline{N}\rangle\overline{L}\cdot\langle N\rangle\overline{L}\cdot\langle T\overline{M}\overline{L}N\rangle\overline{L}\cdot\langle T^2\rangle\overline{L}=1.$$

∎

Applying Lemma 7(b), the left side of lantern law can be rewritten in the following form:

$$\langle T\rangle L\cdot\langle N^3\bar{L}\bar{M}L^2TN\rangle L\cdot\langle\bar{N}TN\rangle L$$

$$\cdot\langle\bar{N}\rangle\bar{L}\cdot\langle N\rangle\bar{L}\cdot\langle T\bar{M}\bar{L}N\rangle\bar{L}\cdot\langle T^2\rangle\bar{L}$$

$$= N^3\bar{M}TKML_2\bar{T}MLN^2TKML_2\bar{T}N\cdot\bar{N}\bar{L}N^2\bar{L}\bar{N}T\bar{M}\bar{L}\bar{M}\bar{L}_2\bar{K}LM\bar{T}\bar{L}_3$$

$$\text{by } L^2\leftrightarrow TN\bar{T},\, TL\bar{T}$$

$$= \bar{M}_2K_{13}ML_3N^3MLN^2K_{23}M_2L_3\bar{L}\bar{M}_2N\bar{M}_2\bar{L}_2\bar{M}_2\bar{L}_3\bar{K}_{23}L_2M_2\bar{L}_3$$

$$\text{by } \bar{M}_2 = N^2\bar{L}\bar{N}^2$$

$$= \langle\bar{M}_2\bar{L}_2\rangle\left\{L_2K_{13}\bar{N}LN^3LNLN^2K_{23}\bar{L}N\bar{M}_2\bar{L}_2\bar{M}_2\bar{K}_{23}\right\}$$

$$= \langle\bar{M}_2\bar{L}_2\rangle\left\{K_{13}L_2\bar{N}LN^2\bar{L}\bar{N}\bar{L}\bar{N}\bar{L}NK_{23}\bar{L}N\bar{M}_2\bar{L}_2\bar{M}_2\bar{K}_{23}\right\}$$

$$\text{by } (LN)^2 = (\bar{N}\bar{L})^3$$

$$= \langle\bar{M}_2\bar{L}_2\rangle\left\{K_{13}\bar{K}_{12}\bar{L}\bar{M}K_{23}\bar{L}N\bar{M}_2\bar{L}_2\bar{M}_2\bar{K}_{23}\right\}$$

$$= \left\langle\bar{M}_2\bar{L}_2\right\rangle\left\{K_{13}\bar{K}_{12}K_{23}\bar{M}\bar{L}\bar{M}\bar{L}_2N\bar{L}_2\bar{M}_2\bar{K}_{23}\right\}$$

$$\text{by } \bar{L}\bar{M}\bar{L} = \bar{M}\bar{L}\bar{M},\ N\bar{M}_2 = N^3\bar{L}\bar{N}^2 = \bar{L}_2N$$

$$= \left\langle\bar{M}_2\bar{L}_2\right\rangle\left\{K_{13}\bar{K}_{12}K_{23}\bar{L}_2\bar{M}\bar{L}\bar{M}N\bar{L}_2\bar{M}_2\bar{K}_{23}\right\}$$

$$= \left\langle\bar{M}_2\bar{L}_2\right\rangle\left\{K_{13}\bar{K}_{12}K_{23}\bar{L}_2\bar{M}(\bar{L}\bar{N})^4\bar{N}\bar{K}_{23}\right\}$$

$$= \left\langle\bar{M}_2\bar{L}_2\right\rangle\left\{K_{13}\bar{K}_{12}K_{23}\bar{L}_2\bar{M}NL\bar{N}\bar{K}_{23}\right\} \qquad \text{by } (\bar{L}\bar{N})^4 = NL$$

$$= \left\langle\bar{M}_2\bar{L}_2\right\rangle\left\{K_{13}\bar{K}_{12}K_{23}K_{12}\bar{K}_{23}\right\}.$$

Therefore, the lemma is clear. ■

LEMMA 9. *The defining relation*

$$L \leftrightarrow \left\langle \bar{N}T\bar{N}(\bar{L}N)^2 \right\rangle L$$

can be replaced by the following relation:

$$L \leftrightarrow \langle N^2\bar{T}N \rangle L.$$

Proof. Indeed, since

$$\left\langle \bar{N}T\bar{N}(\bar{L}N)^2 \right\rangle L = \left\langle \bar{N}T\bar{N}(\bar{N}L)^3 \right\rangle L$$

$$\text{by } (\bar{L}N)^{10} = 1,\ L \leftrightarrow (\bar{N}L)^5$$

$$= \bar{N}T\bar{N}^2L\bar{N}L\bar{N}LN\bar{L}N\bar{L}N^2\bar{T}N$$

$$= \bar{N}TN^2 \cdot N^2L\bar{N}^2 \cdot NL\bar{N}LN \cdot \bar{L}N\bar{L}N^2\bar{T}N$$

$$= \bar{N}TN^2\bar{L}N^2L\bar{N}L\bar{N}LN^2\bar{L}N^2\bar{T}N,$$

$$\text{by } L \leftrightarrow NL\bar{N}LN,\ L \leftrightarrow N^2L\bar{N}^2$$

$$= \bar{N}TN^2\bar{L}N\bar{L}NL\bar{N}L\bar{N}LN^4\bar{T}N$$

$$= \bar{N}T^2\bar{M}\bar{L}\bar{T}^2 \cdot T\bar{N}^2LN^2\bar{T} \cdot T^2LM\bar{T}^2N$$

$$= T^2\bar{M}\bar{L}\bar{T}^2 \cdot \bar{N}T\bar{N}^2LN^2\bar{T} \cdot NT^2LM\bar{T}^2,$$

we may replace the relation $L \leftrightarrow \langle \bar{N}T\bar{N}(\bar{L}N)^2 \rangle L$ by the formula

$$L \leftrightarrow \langle \bar{N}T\bar{N}^2 \rangle L,$$

or equivalently,

$$L \leftrightarrow \langle N^2\bar{T}N \rangle L. \qquad \blacksquare$$

Now we have Theorem 3 proved.

REFERENCES

1 J. S. Birman, On Siegel's modular group, *Math. Ann.* 191:59–68 (1971).
2 N. Lu, On the mapping class group of the torus, *Stud. Cerc. Mat.* 40:137–141 (1988).
3 N. Lu, On the mapping class group of the closed orientable surface of genus two, *Topology Proc.* 13:249–291 (1988).
4 N. Lu, On the mapping class group of the closed orientable surface, *Topology Proc.* 13:293–324 (1988).
5 W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Publications, Inc., New York, 1966, 1976.
6 J. Powell, Two theorems on the mapping class group of a surface, *Proc. Amer. Math. Soc.* 68:347–350 (1977).

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